ON DIRECT SUM DECOMPOSITIONS OF HESTENES ALGEBRAS

BY

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ABSTRACT

In a *-linear Hestenes algebra, the elements with *-reciprocals are characterized by means of certain direct sum decompositions of the algebra.

Introduction. The generalizations of the concepts of Hermitian and normal matrices, and of self-adjoint and normal closed dense operators on a Hilbert space [2, 3] and of a spectral theory for such operators,[2] led Hestenes to introduce a ternary algebra with an involution [4], as the natural framework for these developments. In the detailed study of this algebra in [4], the concept of a *-reciprocal[3] plays a central role; as in [5] it is shown to be closely related to a minimal polynomial in ternary powers. Thus the existence of the latter is sufficient for that of the former ([4] theorem 11.2).

In this note we invoke suitable regularity conditions, rather than minimum polynomials, to characterize the elements of a *-linear Hestenes algebra $\mathscr{A}$ which have *-reciprocals (theorem 5(b) below). We relate the *-reciprocal to certain direct sum decompositions of $\mathscr{A}$, which are of independent interest. Other decompositions of $\mathscr{A}$ were given in [4, §7].

1. Let $\mathscr{A}$ be a *-linear ternary algebra in the sense of Hestenes [4], and let $\mathscr{A}^*$, with $A^*BC^* = (CB^*A)^*$ as a triple product, be the *-linear ternary algebra conjugate to $\mathscr{A}$, e.g. [4, p. 141]. Following [4] we denote by a prime the *-reciprocal of the element in question, whenever it exists, i.e. A' is the *-reciprocal of $A \in \mathscr{A}$ ([4, p. 150]). By $\mathscr{A} = \mathscr{B} \oplus \mathscr{C}$ we mean that $\mathscr{A}$ is the *direct sum* of the classes $\mathscr{B}, \mathscr{C}$; i.e. that every $A \in \mathscr{A}$ can be uniquely expressed as $A = B + C$ with $\mathscr{B} \in \mathscr{B}$, $C \in \mathscr{C}$. For an ordered pair $\{A, B\}$ of element of $\mathscr{A}$ we define the classes:

$$R\{A, B\} = \{C \in \mathscr{A} : C = AU^*B \text{ for some } U \in \mathscr{A}\}$$
$$N\{A^*, B^*\} = \{C \in \mathscr{A} : A^*CB^* = 0\}[4]$$

Received September 9, 1963.

(1) Research partly supported by the Office of Naval Research, contract Nonr-1228(10) project NR 047–021, and by the National Science Foundation, project B-14102.

(2) See the references in [4], pp. 139 and 140.

(3) For closed dense linear operators on a Hilbert space the *-reciprocal coincides with the adjoint of the generalized inverse, e.g. [5], [1].

(4) 0 denotes the null element in both $\mathscr{A}$, $\mathscr{A}^*$

respectively called the *range of* $\{A, B\}$, the *null space of* $\{A^*, B^*\}$. An element $A \in \mathscr{A}$ will be called *regular* if $A = AP^*A$ for some $P \in \mathscr{A}$; and *R-regular* if it is regular and in addition $\mathscr{A} = R\{A, A\} \oplus N\{A^*, A^*\}$. The analogous definitions for $\mathscr{A}^*$ are clear. All the results below have analogous counterparts for the conjugate algebra, which will be omitted.

2. Let A, B be any elements of $\mathscr{A}$.
   (a) $R\{A, B\}$ is a ternary subalgebra of $\mathscr{A}$.
   (b) $N\{A^*, B^*\}$ is a linear subspace of $\mathscr{A}$.
   (c) $R\{A^*, B^*\} = \{R\{B, A\}\}^*$, $N\{A, B\} = \{N\{B^*, A^*\}\}^*$
   (d) $R(A, A\} \cap N\{A^*, A^*\} = \{0\}$, the set consisting of the null element.
   (e) If the *-reciprocals A', B' exist, then $N\{A, B\} = N\{A', B'\}$.

Proof. Parts (a) (b) and (c) are obvious.

(d) Let $X \in R\{A, A\} \cap N\{A^*, A^*\}$, i.e., $X = AU^*A$ for some $U \in \mathscr{A}$, and $A^*AU^*AA^* = 0$. By [4, Theorem 4.1], this is equivalent to $A^*UA^* = 0$, thus $X = 0$.

(e) Follows from:

$$AX^*B = AA^*(A'X^*B')B^*B$$

$$A'X^*B' = A'A'^*(AX^*B)B'^*B' \text{ for all } X \in \mathscr{A}$$

3. Theorem. *Let A, B be any elements of $\mathscr{A}$, with *-reciprocals A', B'.*
   (*a*) *The equation*

$$A^*XB^* = C^*, \tag{3.1}$$

C^* given in $\mathscr{A}^*$, is solvable (i.e., $C^* \in R\{A^*, B^*\}$) if and only if:

$$A^*A'C^*B'B^* = C^*. \tag{3.2}$$

(b) $\mathscr{A}^* = R\{A^*, B^*\} \oplus N\{A, B\}$.

(c) *$N\{A^*, B^*\}$ is the central orthogonal complement* ([4, p. 146]) *of $R\{A, B\}$.*

Proof.

(a) As in [5], theorem 2.

(b) Write any element $X^* \in \mathscr{A}^*$ as

$$X^* = Y^* + Z^* \tag{3.3}$$

where

$$Y^* = A^*A'X^*B'B^* \tag{3.4}$$

is in $R\{A^*, B^*\}$.

That $Z^* \in N\{A, B\}$ follows from:

$$AZ^*B = A(X^* - Y^*)B = AX^*B - AA^*A'X^*B'B^*B = 0.$$

It remains to show that $R\{A^*, B^*\} \cap N\{A, B\} = \{0\}$.

Suppose $X^* \in R\{A^*, B^*\} \cap N\{A, B\}$. Then

$X^* = A^*A'X^*B'B^*$ by (a)

and $X^* \in N\{A', B'\}$ by 2(e),

thus $X^* = 0$.

(c) If $Z \in N\{A^*, B^*\}$ then

(3.5) $$AX^*BZ^*AX^*B = 0, \text{ for all } X \in \mathscr{A}.$$

Thus $N\{A^*, B^*\}$ is contained in the central orthogonal complement of $R\{A, B\}$. Conversely, suppose (3.5) holds for all $X \in \mathscr{A}$ and set $X = BZ^*A$. Then X^* and X^*XX^* are in $R\{A^*, B^*\}$. From (3.5), on the other hand, $X^*XX^* \in N\{A, B\}$. Therefore $X^*XX^* = 0$ and $X^* = 0$, i.e., $Z \in N\{A^*, B^*\}$.

REMARKS.

(a) In the matrix case part (a) was given by Penrose ([5, Theorem 2]), and part (b) is due to Ben-Israel–Charnes ([1, Theorem 20]).

(b) Part (b) is analogous to the Fredholm alternative theorem.

4. THEOREM. *Let A be any element of $\mathscr{A}$.*

(*a*) *The equation*:

(4.1) $$A = XA^*A$$

when solvable, has a unique solution in $R\{A, A\}$.

(*b*) *If A has a *-reciprocal A', then A' is the unique solution of* (4.1) *which lies in $R\{A, A\}$.*

Proof.

(a) A solution X of (4.1) is in $N\{A^*, A^*\}$ if and only if $A = 0$. This follows from the facts:

(i) $A = 0$ if and only if $AA^*A = 0$,

(ii) $A^*XA^* = 0$ if and only if $AA^*XA^*A = 0$.

Suppose now that (4.1) has two solution X_1, X_2 in $R\{A, A\}$. Then

$$AA^*(X_1 - X_2)A^*A = 0$$

which is equivalent to $(X_1 - X_2) \in N\{A^*, A^*\}$. Therefore $X_1 = X_2$.

(b) By [4], equation 6.1, A' satisfies (4.1) and is in $R\{A, A\}$. Uniqueness follows from (a).

REMARK. In the matrix case equation (4.1) is the normal equation and part (b) is the statement that $A^{+}b$ is the least squares solution of $Ax = b$, (see, e.g. [6]).

5. THEOREM. *Let A be any element of $\mathscr{A}$.*

(a) *If $A = AP^*A$ for some $P \in \mathscr{A}$ then $X \in R\{A, A\}$ if and only if*

$$X = AP^*XP^*A. \tag{5.1}$$

(b) *The *-reciprocal A' exists if and only if A is R-regular.*

(c) *A is R-regular if and only if $A = AP^*A$ and A permutes with P.*

Proof.

(a) As in [5, Theorem 2], noting that only the regularity of A is used.

(b) If A' exists then

$$A = AA'^*A \text{ and}$$

$$\mathscr{A} = R\{A, A\} \oplus N\{A^*, A^*\}, \text{ by 3(b).} \tag{5.2}$$

Thus A is R-regular with $P = A'$.

Conversely suppose that

(5.3) $A = AP^*A$ for some $P \in \mathscr{A}$ and that (5.2) holds. By (5.2) there is a unique P in $R\{A, A\}$ satisfying (5.3); we show now that this P is the *-reciprocal of A. By [4, theorem 6.1] it suffices to show that:

$$A = PA^*A \text{ and} \tag{5.4}$$

$$P = PP^*A. \tag{5.5}$$

To prove (5.4) consider $D = A - PA^*A$, which is, by definition*, in $R\{A, A\}$. From (5.3) we conclude that $D \in N\{A^*, A^*\}$ and thus that $D = 0$.

To prove (5.5) we write

$$\begin{aligned} PP^*A &= (AP^*PP^*A)P^*A, && \text{by (a)} \\ &= AP^*PP^*A, && \text{by (5.3)} \\ &= P, && \text{by (a).} \end{aligned}$$

(c) Let (5.3) hold for some P which permutes with A. For any $X \in \mathscr{A}$, the element AP^*XP^*A is in $R\{A, A\}$ and the element $X - AP^*XP^*A$ is in $N\{A^*, A^*\}$, since

$$A^*(X - AP^*XP^*A)A^* = A^*XA^* - A^*AP^*XP^*AA^*$$

and the conditions on P. The proof of (5.2) is now completed by using 2(d). Thus A is R-regular. Conversely, if A is R-regular then, by (b) and [4, theorem 6.1], there is a $P \in \mathscr{A}$, namely $P = A'$, which permutes with A and satisfies (5.3).

* Since A is R-regular, and $A = O \Leftrightarrow A^*AA^* = O$, it follows that $A \in R[A,A]$.

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